Second- and fourth-order invariants on curved manifolds with torsion

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Second- and fourth-order invariants on curved manifolds with torsion

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Abstract. The second- and fourth-order scalar invariants on curved manifolds with torsion are tabulated. Relations among these invariants arising from the Bianchi, cyclic and Ricci identities are presented. The number of invariants is seen to be greatly reduced when the torsion tensor is taken to be totally anti-symmetric. The large number of fourth-order invariants indicates that, in general, the expressions normally obtained during the quantisation of a gravitational theory including torsion will be extremely cumbersome. There are certain exceptions to this statement, however. The most famous example of this is supergravity theory.

1. Introduction

Einstein's general theory of relativity has been a very successful theory. However, this has not stopped a multitude of efforts to construct a better theory. At the classical level we might hope to obtain a simpler theory or perhaps a theory in closer agreement with experiment. It could also be that an improved classical theory will lead to the much-dreamed-of renormalisable quantum theory of gravity. Finally, it may be possible to unify gravity with the other forces of nature only with some expanded version of general relativity.

There are several fairly simple ways to change standard general relativity. One is the addition of a cosmological term to the usual Einstein action. (See Christensen and Duff (1979b) for a recent discussion of this.) There does not appear to be much experimental need for a cosmological constant, nor does it seem to be of any help in the renormalisation problem. It is however necessary in certain attempts at unification, such as the extended supergravity models (Deser and Zumino 1976, Freedman *et al* 1976). Some calculations in these theories with a cosmological constant are difficult, if not impossible, due to the non-existence of a well-defined S matrix, but others, such as the computation of one-loop counterterms and gravitational anomalies, are quite easy (Christensen and Duff 1979b, Christensen *et al* 1980a).

A second generalisation of standard general relativity can be made by introducing a non-zero torsion into the theory. Most of the work on torsion theories has been at the classical level. (See Hehl *et al* (1976) for a review.) In this paper we ask the question: What will the one-loop structure of a quantised gravitational theory with torsion be? We will not do any actual loop calculation to answer this question, since it is already well known that at the one-loop level the infinity structure of the functional integral is built from the scalar invariants of order four in derivatives of the metric. *A priori* then, the one-loop counterterm will be a linear combination of these invariants. Thus we will need to know what the fourth-order scalar invariants are and what relationships there may be between them. This is the obvious first step one must take before proceeding with any actual calculation.

In § 2 we define our notation. We will be using Misner, Thorne and Wheeler conventions wherever possible. Section 3 consists of a list of all possible second- and fourth-order scalar invariants constructed from the torsion tensor, Riemann tensor and their derivatives. Relationships between these invariants obtained from the Bianchi, cyclic and Ricci identities are presented in § 4 along with those invariants which remain if we assume the torsion tensor to be totally anti-symmetric. The final section contains a discussion of quantum calculations in torsion theories.

2. Notation

We will consider a manifold with metric $g_{\alpha\beta}$ and connection $\Gamma^{\gamma}_{\alpha\beta}$. In general relativity the connection is taken to be symmetric in the lower indices. One obvious way to build a new theory is to relax this restriction and allow the connection to have an anti-symmetric part. The 'metric compatibility' condition

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^{\rho}_{\alpha\gamma} g_{\rho\beta} - \Gamma^{\rho}_{\beta\gamma} g_{\alpha\rho} = 0 \tag{2.1}$$

used in the standard way gives us

$$\left\{ {\gamma \atop \alpha\beta} \right\} = \tfrac{1}{2}(\Gamma^{\gamma}{}_{\alpha\beta} + \Gamma^{\gamma}{}_{\beta\alpha}) + \tfrac{1}{2}T^{\gamma}{}_{\beta\alpha} + \tfrac{1}{2}T^{\gamma}{}_{\alpha\beta}, \tag{2.2}$$

where $\left\{ {\gamma \atop \alpha\beta} \right\}$ is the usual Christoffel symbol, and

$$T_{\alpha\beta}{}^{\gamma} = \Gamma^{\gamma}{}_{\alpha\beta} - \Gamma^{\gamma}{}_{\beta\alpha} \tag{2.3}$$

is the torsion tensor. (Some notations put an overall factor of $\frac{1}{2}$ on the right-hand side of (2.3).) This tensor is obviously anti-symmetric in its first two indices. Equation (2.2) tells us that

$$\Gamma^{\gamma}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} - \tfrac{1}{2}T^{\gamma}{}_{\beta\alpha} - \tfrac{1}{2}T^{\gamma}{}_{\alpha\beta} + \tfrac{1}{2}T_{\alpha\beta}{}^{\gamma}. \tag{2.4}$$

We note that, when the torsion vanishes, the connection is the Christoffel symbol. We also note that the symmetric part of the connection is equal to the Christoffel symbol only if the torsion is anti-symmetric in all of its indices.

The Riemann tensor is defined by

$$R^{\mu}{}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\tau}_{\alpha\gamma}\Gamma^{\mu}_{\tau\beta} - \Gamma^{\tau}_{\alpha\beta}\Gamma^{\mu}_{\tau\gamma} \tag{2.5}$$

as usual. It has the symmetries

$$R_{\alpha\mu\beta\gamma} = -R_{\mu\alpha\beta\gamma}, \qquad R_{\mu\alpha\gamma\beta} = -R_{\mu\alpha\beta\gamma}, \tag{2.6}$$

but, because the connection is no longer symmetric, we will *not* be able to use

$$R_{\beta\gamma\mu\alpha} = R_{\mu\alpha\beta\gamma}. \tag{2.7}$$

From the Riemann tensor we can build the Ricci tensor

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}, \tag{2.8}$$

which, due to the missing symmetry (2.7), is *not* symmetric, i.e.

$$R_{\beta\alpha} \neq R_{\alpha\beta}. \tag{2.9}$$

Finally, the Riemann scalar is

$$R = R^{\alpha}{}_{\alpha}. \tag{2.10}$$

Throughout this paper we will write either $_{;\alpha}$ or ∇_α to represent the covariant derivative with connection $\Gamma^{\gamma}_{\alpha\beta}$.

3. Invariants

In the quantum theory of gravity with torsion we are concerned at present with the tree- and one-loop levels. In these two cases we will need to know the second- and fourth-order invariants constructed from the torsion tensor $T_{\alpha\beta\gamma}$, which is of order one, the covariant derivative ∇_α, also of order one, and the Riemann tensor $R_{\alpha\beta\gamma\delta}$, an order-two object. In the lists that follow we use only the anti-symmetry on the first two indices of the torsion tensor and the anti-symmetry properties of the Riemann tensor. Only in the next section will we use further symmetry relationships or identities.

At order two we have three tensors from which we can build scalar invariants:

$$T_{\alpha\beta\gamma;\delta}, \qquad T_{\alpha\beta\gamma}T_{\delta\epsilon\kappa}, \qquad R_{\alpha\beta\gamma\delta}.$$

We see that there are only five such invariants. We would expect there to be many more fourth-order invariants, and indeed there are—194 in all! We construct scalars from the four-order tensors:

$$R_{\alpha\beta\gamma\delta;\epsilon\kappa}, \quad R_{\alpha\beta\gamma\delta}R_{\epsilon\kappa\rho\tau}, \quad T_{\alpha\beta\gamma;\delta\epsilon\kappa}, \quad T_{\alpha\beta\gamma;\delta}R_{\epsilon\kappa\rho\tau}, \quad T_{\alpha\beta\gamma;\delta}T_{\epsilon\kappa\rho;\tau},$$

$$T_{\alpha\beta\gamma}R_{\delta\epsilon\kappa\rho;\tau}, \quad T_{\alpha\beta\gamma}T_{\delta\epsilon\kappa}R_{\rho\tau\sigma\mu}, \quad T_{\alpha\beta\gamma}T_{\delta\epsilon\kappa;\rho\tau}, \quad T_{\alpha\beta\gamma}T_{\delta\epsilon\kappa}T_{\rho\tau\sigma;\mu},$$

$$T_{\alpha\beta\gamma}T_{\delta\epsilon\kappa}T_{\rho\tau\sigma}T_{\mu\nu\lambda}.$$

Using a diagrammatic technique which we shall not discuss here, we obtain the invariants:

Table of second-order invariants

$I(1) = T_{\alpha\beta}{}^{\alpha}{}_{;}{}^{\beta}$ $\quad I(2) = T_{\alpha\beta}{}^{\alpha}T_{\gamma}{}^{\beta\gamma}$ $\quad I(3) = T_{\alpha\beta\gamma}T^{\alpha\beta\gamma}$

$I(4) = T_{\alpha\beta\gamma}T^{\alpha\gamma\beta}$ $\quad I(5) = R$

Table of fourth-order invariants

$I(6) = R_{;\alpha}{}^{\alpha}$
$I(7) = R_{\alpha\beta;}{}^{\alpha\beta}$
$I(8) = R_{\alpha\beta;}{}^{\beta\alpha}$
$I(9) = R^2$
$I(10) = R_{\alpha\beta}R^{\alpha\beta}$
$I(11) = R_{\alpha\beta}R^{\beta\alpha}$
$I(12) = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$
$I(13) = R_{\alpha\beta\gamma\delta}R^{\alpha\delta\gamma\beta}$
$I(14) = T_{\alpha\beta}{}^{\alpha}{}_{;}{}^{\beta}{}_{\gamma}{}^{\gamma}$
$I(15) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}{}^{\beta\gamma}$

$I(16) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}{}^{\gamma\beta}$

$I(17) = T_{\alpha\beta\gamma;}{}^{\alpha\beta\gamma}$

$I(18) = T_{\alpha\beta\gamma;}{}^{\alpha\gamma\beta}$

$I(19) = T_{\alpha\beta\gamma;}{}^{\gamma\alpha\beta}$

$I(20) = T_{\alpha\beta}{}^{\alpha}{}_{;}{}^{\beta}R$

$I(21) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}R^{\beta\gamma}$

$I(22) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}R^{\gamma\beta}$

$I(23) = T_{\alpha\beta\gamma;}{}^{\alpha}R^{\beta\gamma}$

$I(24) = T_{\alpha\beta\gamma;}{}^{\alpha}R^{\gamma\beta}$

$I(25) = T_{\alpha\beta\gamma;}{}^{\gamma}R^{\alpha\beta}$

$I(26) = T_{\alpha\beta\gamma;\delta}R^{\alpha\beta\gamma\delta}$

$I(27) = T_{\alpha\beta\gamma;\delta}R^{\alpha\gamma\beta\delta}$

$I(28) = T_{\alpha\beta\gamma;\delta}R^{\alpha\delta\beta\gamma}$

$I(29) = T_{\alpha\beta\gamma;\delta}R^{\gamma\delta\alpha\beta}$

$I(30) = T_{\alpha\beta}{}^{\alpha}{}_{;}{}^{\beta}T_{\gamma\delta}{}^{\gamma}{}_{;}{}^{\delta}$

$I(31) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}T_{\delta}{}^{\beta\delta}{}_{;}{}^{\gamma}$

$I(32) = T_{\alpha\beta}{}^{a}{}_{;\gamma}T_{\delta}{}^{\gamma\delta}{}_{;}{}^{\beta}$

$I(33) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}T_{\delta}{}^{\beta\gamma}{}_{;}{}^{\delta}$

$I(34) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}T_{\delta}{}^{\gamma\beta}{}_{;}{}^{\delta}$

$I(35) = T_{\alpha\beta}{}^{\alpha}{}_{;\gamma}T^{\beta\gamma}{}_{\delta;}{}^{\delta}$

$I(36) = T_{\alpha\beta\gamma;}{}^{\alpha}T_{\delta}{}^{\beta\gamma}{}_{;}{}^{\delta}$

$I(37) = T_{\alpha\beta\gamma;}{}^{\alpha}T_{\delta}{}^{\gamma\beta}{}_{;}{}^{\delta}$

$I(38) = T_{\alpha\beta\gamma;}{}^{\alpha}T^{\beta\gamma}{}_{\delta;}{}^{\delta}$

$I(39) = T_{\alpha\beta\gamma;}{}^{\gamma}T^{\alpha\beta}{}_{\delta;}{}^{\delta}$

$I(40) = T_{\alpha\beta\gamma;\delta}T^{\alpha\beta\gamma}{}_{;}{}^{\delta}$

$I(41) = T_{\alpha\beta\gamma;\delta}T^{\alpha\beta\delta}{}_{;}{}^{\gamma}$

$I(42) = T_{\alpha\beta\gamma;\delta}T^{\alpha\gamma\beta}{}_{;}{}^{\delta}$

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$I(45) = T_{\alpha\beta\gamma;\delta}T^{\alpha\delta\gamma}{}_{;}{}^{\beta}$

$I(46) = T_{\alpha\beta\gamma;\delta}T^{\gamma\delta\alpha}{}_{;}{}^{\beta}$

$I(47) = T_{\alpha\beta}{}^{\alpha}R_{;}{}^{\beta}$

$I(48) = T_{\alpha\beta}{}^{\alpha}R_{\gamma}{}^{\beta}{}_{;}{}^{\gamma}$

$I(49) = T_{\alpha\beta}{}^{\alpha}R^{\beta}{}_{\gamma;}{}^{\gamma}$

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$I(57) = T_{\alpha\beta}{}^{\alpha}T_{\gamma}{}^{\beta\gamma}R$

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$I(78) = T_{\alpha\beta}{}^{\alpha}T_{\gamma\delta\epsilon}R^{\delta\epsilon\beta\gamma}$

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$I(172) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta}{}^{\gamma}T_{\epsilon}{}^{\beta}{}_{\kappa}T^{\kappa\delta\epsilon}$
$I(173) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta}{}^{\gamma}T_{\epsilon\kappa}{}^{\epsilon}T^{\kappa\beta\delta}$
$I(174) = T_{\alpha\beta\gamma}T^{\gamma}{}_{\delta}{}^{\alpha}T_{\epsilon}{}^{\beta}{}_{\kappa}T^{\kappa\delta\epsilon}$
$I(175) = T_{\alpha\beta\gamma}T^{\gamma}{}_{\delta}{}^{\alpha}T_{\epsilon\kappa}{}^{\beta}T^{\epsilon\delta\kappa}$
$I(176) = T_{\alpha\beta\gamma}T^{\gamma}{}_{\delta}{}^{\alpha}T_{\epsilon\kappa}{}^{\epsilon}T^{\kappa\beta\delta}$
$I(177) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta}{}^{\beta}T_{\epsilon\kappa}{}^{\gamma}T^{\epsilon\delta\kappa}$
$I(178) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta}{}^{\beta}T_{\epsilon\kappa}{}^{\epsilon}T^{\kappa\gamma\delta}$
$I(179) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta}{}^{\beta}T_{\epsilon\kappa}{}^{\epsilon}T^{\gamma\delta\kappa}$
$I(180) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\beta\delta}T^{\kappa\gamma\epsilon}$
$I(181) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\beta\epsilon}T^{\kappa\gamma\delta}$
$I(182) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\beta\delta}T^{\kappa\epsilon\delta}$
$I(183) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\beta\epsilon}T^{\kappa\delta\gamma}$
$I(184) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\beta\delta}T^{\kappa\epsilon\kappa}$
$I(185) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\beta\epsilon}T^{\gamma\delta\kappa}$
$I(186) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\delta\beta}T^{\gamma\epsilon\kappa}$
$I(187) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\epsilon\beta}T^{\gamma\delta\kappa}$
$I(188) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\gamma\delta}T^{\beta\epsilon\kappa}$
$I(189) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\gamma\epsilon}T^{\beta\delta\kappa}$
$I(190) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\delta\gamma}T^{\beta\epsilon\kappa}$
$I(191) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T_{\kappa}{}^{\epsilon\gamma}T^{\beta\delta\kappa}$
$I(192) = T_{\alpha\beta\gamma}T_{\delta\epsilon}{}^{\alpha}T_{\kappa}{}^{\beta\delta}T^{\gamma\epsilon\kappa}$
$I(193) = T_{\alpha\beta\gamma}T_{\delta\epsilon}{}^{\alpha}T_{\kappa}{}^{\delta\beta}T^{\gamma\epsilon\kappa}$
$I(194) = T_{\alpha\beta\gamma}T_{\delta\epsilon}{}^{\alpha}T_{\kappa}{}^{\gamma\delta}T^{\beta\epsilon\kappa}$
$I(195) = T_{\alpha\beta\gamma}T_{\delta\epsilon}{}^{\alpha}T_{\kappa}{}^{\delta\gamma}T^{\beta\epsilon\kappa}$
$I(196) = T_{\alpha\beta\gamma}T_{\delta\epsilon}{}^{\alpha}T^{\beta\delta}{}_{\kappa}T^{\gamma\epsilon\kappa}$
$I(197) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta\epsilon}T^{\beta\delta}{}_{\kappa}T^{\gamma\epsilon\kappa}$
$I(198) = T_{\alpha\beta\gamma}T_{\delta\epsilon}{}^{\gamma}T^{\alpha\delta}{}_{\kappa}T^{\beta\epsilon\kappa}$
$I(199) = T_{\alpha\beta\gamma}T^{\alpha}{}_{\delta}{}^{\beta}T_{\epsilon\kappa}{}^{\gamma}T^{\epsilon\delta\kappa}$

4. Relationships between invariants

A quick glance at the list of fourth-order invariants shows us that some of the invariants may be written as a linear combination of others. There are three sources of relations between invariants which we shall consider here: the Ricci identity, the Bianchi identity and the cyclic identity. All these are different in torsion theories from the more familiar ones of torsion-free theories. Using the definitions (2.3) and (2.5) and the definition of the covariant derivative of a tensor

$$V_{\alpha_1\ldots\alpha_N;\beta} = V_{\alpha_1\ldots\alpha_N;\beta} - \Gamma^{\gamma}_{\alpha_1\beta} V_{\gamma\alpha_2\ldots\alpha_n} - \ldots - \Gamma^{\gamma}_{\alpha_N\beta} V_{\alpha_1\ldots\alpha_{n-1}\gamma}, \tag{4.1}$$

it is a straightforward though slightly tedious process to prove:

the Bianchi identity

$$R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\beta\gamma;\alpha} + R_{\mu\nu\gamma\alpha;\beta} = -T_{\alpha\beta}{}^{\rho} R_{\mu\nu\gamma\rho} - T_{\beta\gamma}{}^{\rho} R_{\mu\nu\alpha\rho} - T_{\gamma\alpha}{}^{\rho} R_{\mu\nu\beta\rho}, \tag{4.2}$$

the contracted Bianchi identity

$$R_{\mu\alpha;\gamma} - R_{\mu\gamma;\alpha} + R_{\mu\nu\alpha\gamma;}{}^{\nu} = -T_{\alpha\nu}{}^{\rho} R_{\mu}{}^{\nu}{}_{\gamma\rho} - T_{\nu\gamma}{}^{\rho} R_{\mu}{}^{\nu}{}_{\alpha\rho} + T_{\gamma\alpha}{}^{\rho} R_{\mu\rho}, \tag{4.3}$$

the doubly contracted Bianchi identity

$$R_{;\gamma} - 2R^{\mu}{}_{\gamma;\mu} = T_{\mu\nu\rho} R^{\mu\nu\rho}{}_{\gamma} + 2T_{\gamma\nu\rho} R^{\nu\rho}, \tag{4.4}$$

the Ricci identity

$$V_{\alpha_1\ldots\alpha_N;\gamma\beta} - V_{\alpha_1\ldots\alpha_N;\beta\gamma}$$
$$= T_{\beta\gamma}{}^{\rho} V_{\alpha_1\ldots\alpha_N;\rho} + V^{\rho}{}_{\alpha_2\ldots\alpha_N} R_{\alpha_1\rho\beta\gamma} + \ldots + V_{\alpha_1\ldots\alpha_{N-1}}{}^{\rho} R_{\alpha_N\rho\beta\gamma}, \tag{4.5}$$

the cyclic identity

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta}$$
$$= -T_{\alpha\beta\mu;\gamma} - T_{\beta\gamma\mu;\alpha} - T_{\gamma\alpha\mu;\beta} + T_{\alpha\beta}{}^{\tau} T_{\tau\gamma\mu} + T_{\beta\gamma}{}^{\tau} T_{\tau\alpha\mu} + T_{\gamma\alpha}{}^{\tau} T_{\tau\beta\mu}, \tag{4.6}$$

the contracted cyclic identity

$$R_{\alpha\gamma} - R_{\gamma\alpha} = -T_{\alpha\beta}{}^{\beta}{}_{;\gamma} - T_{\beta\gamma}{}^{\beta}{}_{;\alpha} - T_{\gamma\alpha}{}^{\beta}{}_{;\beta} + T_{\alpha\beta}{}^{\tau} T_{\tau\gamma}{}^{\beta} + T_{\beta\gamma}{}^{\tau} T_{\tau\alpha}{}^{\beta} + T_{\gamma\alpha}{}^{\tau} T_{\tau\beta}{}^{\beta}. \tag{4.7}$$

It is obvious that when $T_{\alpha\beta}{}^{\gamma} = 0$ these identities reduce to the usual identities and symmetries of Einstein's theory.

From the Ricci identity we obtain the relations:

$$I(8) = I(7) + I(50) + I(10) - I(11), \tag{4.8}$$

$$I(15) = I(14) - I(33) + I(102) + I(21) + I(49), \tag{4.9}$$

$$I(16) = I(15) + I(101) + I(21) - I(22), \tag{4.10}$$

$$I(17) = -\tfrac{1}{2}I(41) - \tfrac{1}{2}I(88) - I(25) - I(50) + \tfrac{1}{2}I(29) - \tfrac{1}{2}I(54), \tag{4.11}$$

$$I(18) = I(17) - I(91) - I(23) + I(24), \tag{4.12}$$

$$I(19) = I(18) + I(44) + I(90) - I(23) + I(24) + I(28) + I(51)$$
$$- I(52) - I(53), \tag{4.13}$$

$$I(89) = I(88) + I(140) + 2I(70) - I(61), \tag{4.14}$$

$$I(91) = I(90) - I(137) - I(80) - I(63) + I(71), \tag{4.15}$$

$$I(93) = I(92) - I(130) + I(84) - I(67) + I(72), \tag{4.16}$$

$$I(105) = I(104) - I(135) - I(80) + I(81) - I(62), \tag{4.17}$$

$$I(97)=I(96)+I(145)+I(73)+I(64)-I(65), \tag{4.18}$$

$$I(98)=-\tfrac{1}{2}I(148)-I(66)-\tfrac{1}{2}I(74), \tag{4.19}$$

$$I(100)=I(99)+I(109)-I(60), \tag{4.20}$$

$$I(102)=I(101)-I(120)-I(73), \tag{4.21}$$

$$I(103)=-\tfrac{1}{2}(118)+\tfrac{1}{2}I(74). \tag{4.22}$$

From the Bianchi identities:

$$I(6)=2I(7)+I(26)-I(55)+2I(23)-2I(51), \tag{4.23}$$

$$I(47)=2I(48)-I(76)+2I(64), \tag{4.24}$$

$$I(50)-I(51)-I(53)=I(70)+I(80)+I(63), \tag{4.25}$$

$$2I(52)-I(54)=-2I(71)+I(61). \tag{4.26}$$

From the cyclic identities:

$$I(8)=I(7)+I(16)-I(15)-I(19)-I(103)-I(98), \tag{4.27}$$

$$I(11)=I(10)+I(22)-I(21)-I(25)-I(66), \tag{4.28}$$

$$I(22)=I(21)+I(32)-I(31)-I(35)+I(116), \tag{4.29}$$

$$I(25)=I(35)+\tfrac{1}{2}I(39)+\tfrac{1}{2}I(114), \tag{4.30}$$

$$I(24)=I(23)+I(34)-I(33)-I(38)-I(114), \tag{4.31}$$

$$I(49)=I(48)+I(87)-I(99)+I(97)+I(116)-I(147)-I(152)+I(110), \tag{4.32}$$

$$I(50)=I(101)+\tfrac{1}{2}I(89)+\tfrac{1}{2}I(148)+\tfrac{1}{2}I(118), \tag{4.33}$$

$$I(52)=I(51)+I(103)-I(102)+I(105)-I(133)-I(138)-I(149)-I(119), \tag{4.34}$$

$$I(63)=I(62)+I(120)-I(119)-I(123)-I(179), \tag{4.35}$$

$$I(65)=I(64)+I(110)-I(109)-I(113)+I(163), \tag{4.36}$$

$$I(66)=-I(116)+\tfrac{1}{2}I(114)-I(160), \tag{4.37}$$

$$I(12)-2I(13)=2I(27)-I(29)-2I(81)-I(82), \tag{4.38}$$

$$I(26)-I(27)+I(28)=I(42)-I(46)-I(43)+I(139)-I(135)+I(134), \tag{4.39}$$

$$I(29)-2I(27)=2I(45)-I(40)+2I(136)+I(141), \tag{4.40}$$

$$I(69)-I(70)+I(71)=I(133)-I(128)-I(134)+I(187)-I(175)-I(186), \tag{4.41}$$

$$I(72)-I(69)-I(71)=I(131)-I(130)-I(129)+I(185)+I(191)-I(175), \tag{4.42}$$

$$I(73)-I(75)+I(74)=-I(148)+2I(149)-2I(179), \tag{4.43}$$

$$I(76)+I(73)-I(77)=-I(145)-I(147)-I(151)+I(176), \tag{4.44}$$

$$I(79)-I(80)+I(81)=-I(137)-I(135)+I(139)-I(194)$$
$$-I(199)+I(192), \tag{4.45}$$
$$I(82)-2I(80)=-2I(138)-I(140)-2I(193)+I(167), \tag{4.46}$$
$$I(83)-2I(84)=2I(143)-I(142)-I(196)+I(168)-I(182), \tag{4.47}$$
$$I(55)+I(53)-I(56)=I(50)+I(94)-I(86)+I(126)+I(134)$$
$$-I(135)+I(139). \tag{4.48}$$

Finally, two simple symmetry relations:

$$I(75)=I(73), \tag{4.49}$$
$$I(78)=I(77). \tag{4.50}$$

We have 43 relations, thus reducing the total number of fourth-order invariants from 194 to 151.

If we were to restrict the torsion tensor to be anti-symmetric on all of its indices, we would reduce the number of invariants to 21. All fourth-order invariants may be written as a linear combination of $I(6)$, $I(7)$, $I(9)$, $I(10)$, $I(12)$, $I(25)$, $I(26)$, $I(39)$, $I(40)$, $I(44)$, $I(50)$, $I(58)$, $I(61)$, $I(69)$, $I(70)$, $I(85)$, $I(88)$, $I(128)$, $I(157)$, $I(165)$, $I(180)$.

5. Discussion

We can now see that any calculation involving fourth-order invariants in a torsion theory has the potential for being miserably complicated. Of course, it may be that not all of the invariants will appear in calculations we might perform. Can we determine if quantum field theoretic calculations will really be as cumbersome as the lists of § 3 indicate? The answer is yes.

We consider the calculation of the one-loop counterterms in quantum gravity with torsion. It is by now very well known that these counterterms are determined by the so-called b_4 coefficient in the asymptotic expansion of the heat kernel. (See Christensen and Duff (1979a) for a detailed discussion of this.) This calculation has been done for the case when the torsion is totally anti-symmetric (Goldthorpe 1979). It was found that all of the invariants that could appear, did appear. Preliminary calculations in the more general case indicate that all invariants *will* appear. This clearly does not bode well for a successful quantisation of torsion theories. They are just too messy!

However, there are ways to get around this problem. The most obvious is to put restrictions on the torsion tensor such as the totally anti-symmetric condition. One would have to provide some strong physical reason for doing this of course, and even this strong condition does not simplify our expressions that much. There are still 21 invariants to contend with.

If we think a bit more, we will come upon another way to 'simplify' our problem. Supergravity theories are torsion theories. If we put a powerful restriction like supersymmetry on our thoery, we may be able to eliminate many of the possible counterterms. Those invariants which cannot be put into a supersymmetric combination with others cannot appear in the counterterms. If we study Goldthorpe's calculation, we see that nearly all of the invariants appear multiplied by the unit matrix for each field of spin s considered. Such terms always vanish when these fields are put together into supersymmetric multiplets. Thus we see why supergravity is simple in its one-loop structure.

Finally, we note that we have not mentioned pseudoscalar invariants, which also can play an important role in quantum theories, nor have we discussed consistency conditions relating the quantum fields propagating in the gravitational field to the Riemann tensor and torsion tensor of that gravitational field. (These conditions may be a source of simplification.) We only mention that there are also a huge number of pseudoscalars possible. This fact certainly will compound our problems. These topics will be discussed in a future publication (Christensen *et al* 1980b).

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